AN ANALYTICAL APPROACH TO 2.5D SOUND FIELD REPRODUCTION EMPLOYING CIRCULAR DISTRIBUTIONS OF NON-OMNIDIRECTIONAL LOUDSPEAKERS

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ABSTRACT

We present an approach targeting the physical reproduction of sound fields by means of circular distributions of non-omnidirectional loudspeakers. The approach does not compensate for loudspeaker properties which deviate from certain assumptions as conventional approaches do. It rather allows for the explicit consideration of these properties within some limits which are outlined. The focus of this paper lies on the modal incorporation of the loudspeaker's spatio-temporal transfer function into the loudspeaker driving function.

1. INTRODUCTION

Traditionally, massive-multichannel sound field reproduction approaches like wave field synthesis or higher order Ambisonics assume that the involved secondary sources (i.e. loudspeakers) are omnidirectional. For lower frequencies, this assumption is indeed approximately fulfilled when conventional loudspeakers with closed cabinets are considered. However, for higher frequencies above a few thousand Hertz complex radiation patterns evolve.

A number of approaches based on the theory of multiple-input-multiple-output (MIMO) systems have been proposed in order to compensate for the influence of the reproduction room and the loudspeaker radiation characteristics [1, 2, 3, 4, 5, 6]. Room compensation requires realtime analysis of the reproduced sound field and adaptive algorithms due to the time-variance of room acoustics (e.g. temperature variations). Compensation of the loudspeaker radiation characteristics, such as directivity and frequency response, is less complex since it can be assumed that these characteristics are time-invariant. No adaptation and therefore no real-time analysis is required. However, in order that the radiation characteristics can be compensated for while neglecting the reproduction room, the radiation characteristics of the entire secondary source setup have to be measured under anechoic conditions. When certain physical constraints are accepted, a significant reduction of complexity can be achieved and a continuous formulation of the MIMO approaches can be established. Besides time-invariance, the fundamental physical constraints introduced in the presented approach are:

(1) The secondary source arrangement is circular.

(2) The spatio-temporal transfer function of the secondary sources is rotation invariant. In other words, all individual loudspeakers have to have equal radiation characteristics and have to be orientated towards the center of the secondary source setup.

Requirement (1) can obviously be fulfilled. Preliminary measurements undertaken at Deutsche Telekom Laboratories have shown that typical commercially available loudspeakers with closed cabinets indeed exhibit similar to equal spatio-temporal transfer functions in anechoic condition. This suggests that requirement (2) can also be fulfilled when the acoustical properties of the reproduction room are ignored.

We emphasize that the presented approach is not a compensation for deviations of the loudspeaker radiation characteristics from certain assumptions (e.g. omnidirectionality). It is rather such that the formulation of the approach allows for an explicit consideration thereof. For convenience, we use the term *directivity filter* to refer to that component of the the secondary source driving function which represents the spatio-temporal transfer function of the secondary sources.

The approach treated in this paper has been presented by the authors in [7, 8], whereby formulations were kept general. In this contribution, we investigate in detail the properties of the loudspeaker directivity filters which arise when circular secondary source setups are employed.

2. NOMENCLATURE AND MATHEMATICAL PRELIMINARIES

The following notational conventions are used: For scalar variables lower case denotes the time domain, upper case the temporal frequency domain. Vectors are denoted by lower case boldface. The three-dimensional position vector in Cartesian coordinates is given as $\mathbf{x} = [x\, y\, z]^T$. The Cartesian coordinates are linked to the spherical coordinates via $x = r\cos\alpha\sin\beta$, $y = r\sin\alpha\sin\beta$, and $z = r\cos\beta$. α denotes the azimuth, β the elevation. Confer also to Fig. 1. For functions dependent on spatial coordinates, we use the notations $F(\mathbf{x})$ and $F(r,\alpha,\beta)$ to emphasize a given coordinate system.

The acoustic wavenumber is denoted by k. It is related to the temporal frequency by $k^2 = \left(\frac{\omega}{c}\right)^2$ with ω being the radial frequency and c the speed of sound. Outgoing spherical waves are denoted by $\frac{1}{r}e^{-i\frac{\omega}{c}r}$. i is the imaginary unit ($i = \sqrt{-1}$).

Due to the continuous formulation, we will not refer to loudspeakers but rather to secondary sources and their distributions and also to secondary source driving functions rather than to loudspeaker signals. We employ a number of standard mathematical tools which are defined below.

The Fourier series expansion with respect to α is defined as [9]

$$G(\mathbf{x},\omega) = \sum_{m=-\infty}^{\infty} \mathring{G}_m(r,\beta,\omega)\; e^{im\alpha} \ . \tag{1}$$

A sound field can be described by its spherical harmonics expansion as [9]

$$F(\mathbf{x},\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \breve{F}_n^m(\omega) B_n^m(\mathbf{x},\omega) \ , \tag{2}$$

whereby the basis $B_n^m(\mathbf{x},\omega)$ is the singular basis $S_n^m(\mathbf{x},\omega)$ for purely diverging sound fields, and the regular basis $R_n^m(\mathbf{x},\omega)$ for passing sound fields. Explicitly,

$$S_n^m(\mathbf{x},\omega) = h_n^{(2)}\left(\frac{\omega}{c}r\right) Y_n^m(\alpha,\beta) \ , \tag{3}$$

$$R_n^m(\mathbf{x},\omega) = j_n\left(\frac{\omega}{c}r\right) Y_n^m(\alpha,\beta) \ . \tag{4}$$

$h_n^{(2)}\left(\frac{\omega}{c}r\right)$ denotes the n-th order spherical Hankel function of second kind, $j_n\left(\frac{\omega}{c}r\right)$ the n-th order spherical Bessel function of first

kind [9].
The spherical harmonics $Y_n^m(\alpha,\beta)$ are defined as

$$Y_n^m(\alpha,\beta) = \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}} \cdot P_n^m(\cos\beta)\cdot e^{im\alpha} \ , \qquad (5)$$

with $P_n^m(\cdot)$ denoting the m-th order associated Legendre polynomial of n-th degree.

3. DERIVATION OF THE DRIVING FUNCTION

As outlined by the authors in [7], the physical fundament of the presented approach is the so-called *simple source approach*. The approach can be seen as an analytical formulation of what is known as higher order Ambisonics. The simple source approach for interior problems states that the acoustic field generated by events outside a volume can also be generated by a continuous distribution of secondary simple sources enclosing the respective volume [9].
Sound field reproduction systems are frequently restricted to reproduction in the horizontal plane. The secondary sources are arranged on a circle. In this case, the acoustic scene to be reproduced as well as the receiver positions are bounded to the horizontal plane. In other words, the listener's ears have to be in the same plane like the secondary sources. For this two-dimensional setup the free-field Green's function required by the simple source approach can be interpreted as the spatial transfer function of a line source. This case is treated e.g. in [10].
However, implementations of such systems usually employ loudspeakers with closed cabinets whose spatial transfer function is more accurately modeled by that of a point source. This secondary source type mismatch prevents us from perfectly recreating any source-free sound field inside the secondary source array. We have to expect artifacts. This circumstance is also a well treated problem in WFS [11]. The approach of employing secondary sources which are intended for three-dimensional reproduction in such an imperfect two-dimensional scenario is typically referred to as 2½-dimensional reproduction.
For a circular distribution of secondary sources located in the horizontal plane and centered around the origin of the coordinate system (refer to Fig. 1) the reproduction equation is given by [7]

$$P(\mathbf{x},\omega) = \int_0^{2\pi} D(\alpha_0,\omega)\cdot G(\alpha-\alpha_0,\beta,\omega)\; r_0 \; d\alpha_0 \ , \qquad (6)$$

whereby $P(\mathbf{x},\omega)$ denotes the reproduced sound field, $D(\alpha_0,\omega)$ the driving signal of the secondary source located at the position $\mathbf{x}_0 = r_0\cdot[\cos\alpha_0 \ \sin\alpha_0 \ 0]^T$, and $G(\alpha-\alpha_0,\beta,\omega)$ its spatio-temporal transfer function.
Note that we assume $G(\cdot)$ to be rotation invariant with respect to rotation around the z-axis (we write $G(\alpha-\alpha_0,\beta,\omega)$ instead of $G(\mathbf{x}|\mathbf{x}_0,\omega)$) [9]. This requires that all secondary sources have to exhibit equal spatio-temporal characteristics and have to be orientated towards the center of the secondary source distribution.
To bound our area of interest to the horizontal plane we set the elevation angle β in all position vectors to $\frac{\pi}{2}$ in the remainder of this section.
Equation (6) can be interpreted as a circular convolution and thus the convolution theorem [9]

$$\mathring{P}_m(r,\omega) = 2\pi r_0 \, \mathring{D}_m(\omega)\, \mathring{G}_m(r,\omega) \qquad (7)$$

and therefore

$$\mathring{D}_m(\omega) = \frac{1}{2\pi r_0}\,\frac{\mathring{P}_m(r,\omega)}{\mathring{G}_m(r,\omega)} \qquad (8)$$

featuring the Fourier series expansion coefficients $\mathring{D}_m(\omega)$, $\mathring{P}_m(r,\omega)$, and $\mathring{G}_m(r,\omega)$ applies. $\mathring{G}_m(r,\omega)$ are the Fourier series expansion coefficients of a secondary source positioned at $\mathbf{x}_0 = [r_0 \ 0 \ 0]^T$ with the expansion center at the origin of the coordinate system. Refer to Sec. 5 for a detailed treatment.
Note that (7) only holds for two-dimensional sound fields. Since $\mathring{P}_m(r,\omega)$ and $\mathring{G}_m(r,\omega)$ are generally three-dimensional, (7) only holds in the horizontal plane (i.e. for $\beta = \frac{\pi}{2}$).
From (8) and (1) we can deduce that

$$D(\alpha,\omega) = \frac{1}{2\pi r_0}\sum_{m=-\infty}^{\infty}\frac{\mathring{P}_m(r,\omega)}{\mathring{G}_m(r,\omega)} e^{im\alpha} \ . \qquad (9)$$

We reformulate the spherical harmonics expansion given by equation (2) by exchanging the order of summations to reveal the Fourier series expansion coefficients reading

$$F\left(\mathbf{x}|_{\beta=\frac{\pi}{2}},\omega\right) = \sum_{m=-\infty}^{\infty} e^{im\alpha}\times \\ \times \underbrace{\sum_{n=|m|}^{\infty}\breve{F}_n^m(\omega)\, j_n\left(\frac{\omega}{c}r\right)\sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(0)}_{\mathring{F}_m(r,\omega)} \ . \qquad (10)$$

Introducing the explicit formulation of the Fourier series expansion coefficients $\mathring{P}_m(r,\omega)$ and $\mathring{G}_m(r,\omega)$ given by (10) into (9) yields the explicit driving function $D(\alpha,\omega)$. Analysis of the latter reveals that the radius r does not cancel out. r appears both in the numerator as well as in the denominator in the summation over n in the argument of the spherical Bessel function $j_n\left(\frac{\omega}{c}r\right)$. The driving function is therefore dependent on the receiver position. We thus have to reference the reproduced sound field to a specific radius which is then the only location where the reproduction is exact. In other positions inside the receiver area deviations arise as outlined in Sec. 4.
For convenience we reference the reproduced sound field to the center of the secondary source array ($r = 0$). Refer to section Sec. 4 for a closer look on the properties of the actual reproduced sound field.
At a first stage, setting $r = 0$ in (9) leads to an undefined expression of the form $\frac{0}{0}$ for $n \neq 0$ since spherical Bessel functions of argument 0 equal 0 $\forall n \neq 0$. Application of de l'Hôpital's rule [12] proves that the expression is defined for $r = 0$ and finally yields the driving function $D_{2.5\mathrm{D}}(\alpha,\omega)$ for 2½-dimensional reproduction as [7]

$$D_{2.5\mathrm{D}}(\alpha,\omega) = \frac{1}{2\pi r_0}\sum_{m=-\infty}^{\infty}\frac{\breve{P}_{|m|}^m(\omega)}{\breve{G}_{|m|}^m(\omega)} e^{im\alpha} \ . \qquad (11)$$

Note that the summation over n in (10) reduces to a single addend with $n = |m|$ [7]. The coefficients $\breve{P}_n^m$ and $\breve{G}_n^m(\omega)$ are defined via (2). Equation (11) generally only holds for $|\mathbf{x}| < r_0$ due to the fact that the coefficients $\breve{P}_{|m|}^m(\omega)$ and $\breve{G}_{|m|}^m(\omega)$ are typically derived from interior expansions [9].

4. REPRODUCED SOUND FIELD

We yield the actual sound field reproduced by the circular secondary source distribution by inserting (11) in (6) as [7]

$$P_{2.5\mathrm{D}}(\mathbf{x},\omega) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\breve{P}_{|m|}^m(\omega)\frac{\breve{G}_n^m(\omega)}{\breve{G}_{|m|}^m(\omega)}\; R_n^m(\mathbf{x},\omega) \\ \forall\, r < r_0 \ . \qquad (12)$$

Note that $\breve{P}_{|m|}^m(\omega)$ are the coefficients of the desired sound field.
Refer to [7] for a simulation of a virtual plane wave reproduced by a continuous circular distribution of secondary monopole sources. It can be observed that the reproduced sound field indeed exhibits perfectly plane wave fronts. However, the reproduced sound field experiences an amplitude decay of approximately 3dB with each doubling of the distance to the secondary source distribution along the propagation direction of the virtual plane wave. This amplitude decay is a typical artifact of 2½-dimensional reproduction.

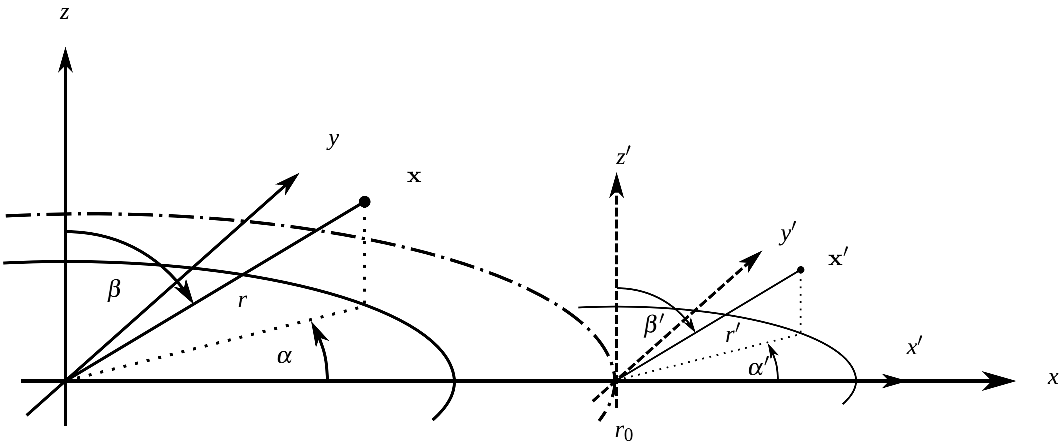

Figure 1: The coordinate systems used in this paper. Primed quantities belong to a local coordinate system at position $\mathbf{x'}_0 = [r_0 \; 0 \; 0]^T$ (see text). The dash-dotted line indicates the secondary source distribution.

5. INCORPORATION OF THE SECONDARY SOURCE DIRECTIVITY COEFFICIENTS

As outlined in Sec. 3, the coefficients $\breve{G}^m_{|m|}(\omega)$ apparent in the driving function (11) describe the spatio-temporal transfer function of a secondary source which is positioned at $\mathbf{x}_0 = [r_0 \; 0 \; 0]^T$ and orientated towards the center of the coordinate system (which coincides with the center of the secondary source distribution). The expansion center is the origin of the coordinate origin. This follows directly from the convolution theorem (7).

However, typical loudspeaker directivity measurements such as [13] yield the coefficients $\breve{G}'^{m'}_{n'}(\omega)$ (see below) of an expansion of the loudspeaker's spatio-temporal transfer function around the acoustical center of the loudspeaker denoted by $\mathbf{x'}_0$. The acoustical center of a loudspeaker is referred to as the position of the latter in the remainder. For convenience, we assume in the following that the loudspeaker under consideration is positioned at $\mathbf{x'}_0 = \mathbf{x}_0 = [r_0 \; 0 \; 0]^T$ and is orientated towards the origin of the global coordinate system. We term the coefficients $\breve{G}'^{m'}_{n'}(\omega)$ *secondary source directivity coefficients*.

We establish a local coordinate system with origin at $\mathbf{x}_0$ and whose axes are parallel to those of the global origin (refer to Fig. 1). Then the spatio-temporal transfer function $G(\mathbf{x'}, \omega)$ of the considered loudspeaker can be described as (refer to (2))

$$G(\mathbf{x'}, \omega) = \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \breve{G}'^{m'}_{n'}(\omega) \, S^{m'}_{n'}(\mathbf{x'}, \omega) \tag{13}$$

with respect to the local coordinate system. Note that

$$\mathbf{x'} = \mathbf{x'}(\mathbf{x}) = \mathbf{x} + \Delta\mathbf{x} \, , \tag{14}$$

with $\Delta\mathbf{x} = [-r_0 \; 0 \; 0]$, $\Delta r = r_0$, $\Delta\alpha = \pi$, and $\Delta\beta = \frac{\pi}{2}$.

In the remainder of this section we demonstrate how the coefficients $\breve{G}^m_{|m|}(\omega)$ required by the secondary source driving function (11) can be yielded from a measurement of the directivity coefficients $\breve{G}'^{m'}_{n'}(\omega)$ by applying appropriate translation operations.

From [14] we know that the translation of the singular part $S^{m'}_{n'}(\mathbf{x'}, \omega)$ of the expansion (13) results in

$$S^{m'}_{n'}(\mathbf{x} + \Delta\mathbf{x}, \omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (S|R)^{m\,m'}_{n\,n'}(\Delta\mathbf{x}, \omega) \, R^m_n(\mathbf{x}) \, . \tag{15}$$

The notation $(S|R)$ indicates that the translation represents a change from a singular basis expansion to a regular basis expansion [9, 14].

Inserting (15) in (13) and re-ordering of the sums reveals the general form of $\breve{G}^m_n(\omega)$ as

$$G(\mathbf{x}, \omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} R^m_n(\mathbf{x}) \times \\ \times \underbrace{\sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \breve{G}'^{m'}_{n'}(\omega) \, (S|R)^{m\,m'}_{n\,n'}(\Delta\mathbf{x}, \omega)}_{= \breve{G}^m_n(\omega)} \, . \tag{16}$$

From the driving function (11) we can deduce that we do not need all coefficients $\breve{G}^m_n(\omega)$ but only $\breve{G}^m_{|m|}(\omega)$

$$\breve{G}^m_{|m|}(\omega) = \sum_{n'=0}^{\infty} \sum_{m'=-n'}^{n'} \breve{G}'^{m'}_{n'}(\omega) \, (S|R)^{m\,m'}_{|m|n'}(\Delta\mathbf{x}, \omega) \, . \tag{17}$$

This facilitates the translation because the sectorial translation coefficients $(S|R)^{m\,m'}_{|m|n'}(\Delta\mathbf{x}, \omega)$ are easier to calculate than the tesseral coefficients $(S|R)^{m\,m'}_{n\;n'}(\Delta\mathbf{x}, \omega)$ [14]. The sectorial translation coefficients can be computed recursively from combinations of the initial value [14]

$$(S|R)^{0\,m'}_{0\,n'}(\Delta\mathbf{x}, \omega) = \sqrt{4\pi} \;\; S^{m'}_{n'}(\Delta\mathbf{x}, \omega) = \\ = (-1)^{m'} \sqrt{(2n'+1)\frac{(n'-m')!}{(n'+m')!}} \, h^{(2)}_{n'}\left(\frac{\omega}{c} r_0\right) P^{m'}_{n'}(0) \tag{18}$$

via the recursion formulae (24) and (25).

It can be shown that the sectorial translation coefficients are of the form

$$(S|R)^{m\,m'}_{|m|n'}(\Delta\mathbf{x}, \omega) = \sum_{l'=0}^{|m|} c^{l',m',n',m} h^{(2)}_{n'-|m|+2l'}\left(\frac{\omega}{c} r_0\right) P^{m'-m}_{n'-|m|+2l'}(0) \, , \tag{19}$$

whereby $c^{l',m',n',m}$ is a real number derived from (18), (24), (25), and (26).

All factors in (19) are always different from zero except for $P^{m'-m}_{n'-|m|+2l'}(0)$ which exhibits zeros wherever $n' - |m| + 2l' + m' - m$ is odd [15]. The latter is equivalent to the case of $n' + m'$ being odd. To take account for this we modify the summations in (17) as

$$\breve{G}^m_{|m|}(\omega) = \sum_{n'=0}^{\infty} \sum_{k'=0}^{n'} \breve{G}'^{2k'-n'}_{n'}(\omega) \, (S|R)^{m,\,2k'-n'}_{|m|,\;n'}(\Delta\mathbf{x}, \omega) \, . \tag{20}$$

This reveals that only the coefficients $\breve{G}'^{2k'-n'}_{n'}(\omega)$ have to be known or measured in order to compute the directivity filter.
Refer to [8] for a simulation of a virtual plane wave reproduced by a continuous distribution of highly directional secondary sources. It can be observed that the desired sound field is indeed perfectly reproduced. Note that the simulation shows a purely two-dimensional scenario. For the present 2½-dimensional problem, the reproduced sound field differs from the desired one as mentioned in section Sec. 4.

6. PROPERTIES OF THE DIRECTIVITY FILTER

6.1 General

As evident from (20), each mode m of the directivity filter is given by a summation over the product of the secondary source directivity coefficients and the translation coefficients. The translation coefficients can be implemented via infinite impulse response filter (IIR) design approaches such as performed in [16]. Alternatively, the digital implementation can be obtained via an appropriate sampling of the analytical mathematical expression (20) which results then in a finite impulse response (FIR) representation.
Due to the fact that the secondary source directivity coefficients are typically yielded from measurements and are modeled as FIR filters, e.g. [13], we propose to also apply the FIR approach on the translation coefficients.
In order that the driving function (11) is defined, neither mode $\breve{G}^m_{|m|}(\omega)$ of the directivity filter may exhibit zeros. From (17) it can be seen that each mode of the directivity filter is given by a summation over all directivity coefficients $\breve{G}'^{2k'-n'}_{n'}(\omega)$ multiplied by the according sectorial translation coefficient $(S|R)^{m,\,2k'-n'}_{|m|,\,n'}(\Delta\mathbf{x},\omega)$.
The translation coefficients are linear combinations of spherical Hankel functions of the same argument but of different orders (refer to (19)). Spherical Hankel functions of different orders are linearly independent [15]. Thus, since spherical Hankel functions do not exhibit zeros, a linear combination of spherical Hankel functions and therefore also the translation coefficients do not exhibit zeros either.
The fact whether the directivity filters are defined is essentially dependent on the properties of the secondary source directivity coefficients $\breve{G}'^{2k'-n'}_{n'}(\omega)$ (refer also to Sec. 6.2).
It has to be noted that the calculation of spherical Hankel functions of high orders and large arguments (i.e. high frequencies or large radii of the secondary source contour) requires high numerical precision. Due to the fact that the directivity filter can be pre-computed there are no performance issues in the calculation.
Secondary source directivity coefficients yielded from measurements of real loudspeakers do not per se result in well-behaved driving functions. Therefore (preferably frequency dependent) regularization such as in [1] has to be applied in order to yield a realizable solution. Contrary to conventional multichannel regularization, the presented approach allows for independent regularization of each mode m of the directivity filter. Thereby, stable modes need not be regularized while the regularization of individual unstable modes can be assumed to be favorable compared to conventional regularization of the entire filter.

6.2 Causality

Spherical Hankel functions of second kind are explicitly given by [14]

$$h^{(2)}_{n'}\left(\frac{\omega}{c}r_0\right) = i^{n'+1}\frac{e^{-i\frac{\omega}{c}r_0}}{\frac{\omega}{c}r_0}\sum_{f'=0}^{n'}\frac{(n'+f')!}{f'(n'-f')!}\left(\frac{1}{2i\frac{\omega}{c}r_0}\right)^{f'} . \quad (21)$$

The exponential term in (21) is independent of the order n' and can be factored out in (20). The exponential term represents a delay in time domain whose duration equals the propagation duration from a secondary source to the center of the secondary source contour. Since the exponential term appears in the numerator of the driving function (11) it turns into an anticipation. In order that the driving function stays causal, this anticipation has to be compensated by an appropriate pre-delay.
Furthermore, the secondary source directivity coefficients $\breve{G}'^{2k'-n'}_{n'}(\omega)$ are generally not minimum phase. The inversion then leads to a filter of infinite length which can not be implemented with an FIR approach. A lack of the minimum phase property can also result in acausal components of the inverse filter. These have to be compensated for via a *modeling delay*. Such a modeling delay is simply an additional delay imposed on the driving function in order to make acausal components causal. Alternatively, the secondary source directivity coefficients can be approximated by minimum phase filters [17].

7. MEASUREMENT OF THE DIRECTIVITY COEFFICIENTS

In this section we investigate the measurement of the spatio-temporal transfer function of a loudspeaker which allows to apply the presented approach. Since we want to compensate only for the horizontal radiation characteristics of the involved secondary sources, we assume that a measurement of the horizontal part of the secondary source's spatio-temporal transfer function is sufficient in order to yield the desired coefficients $\breve{G}'^{2k'-n'}_{n'}(\omega)$. We therefore assume a circular horizontal arrangement of pressure microphones of radius r'_{ref} whose center coincides with the position of the loudspeaker to be measured.
The Fourier series expansion coefficients $\mathring{G}'_m\left(r'|_{r'=r'_{\text{ref}}},\beta'|_{\beta'=\frac{\pi}{2}},\omega\right)$ of the horizontal component of the spatio-temporal transfer function of the loudspeaker can be determined from the microphone signals via [9]

$$\mathring{G}'_m\left(r'|_{r'=r'_{\text{ref}}},\beta'|_{\beta'=\frac{\pi}{2}},\omega\right) = = \frac{1}{2\pi r_0}\int_0^{2\pi} G\left(\mathbf{x}'|_{r'=r'_{\text{ref}},\beta'=\frac{\pi}{2}},\omega\right)\, e^{-im\alpha'}\, d\alpha' . \quad (22)$$

The pressure $G\left(\mathbf{x}'|_{r'=r'_{\text{ref}},\beta'=\frac{\pi}{2}},\omega\right)$ is yielded directly from the microphone signals. The integral in (22) then has to be approximated by an appropriate summation of the measurement points. For convenience, we assume that the microphone spacing is so close that no considerable spatial aliasing occurs.
By exploiting relation (10), the measured coefficients $\mathring{G}'_m\left(r'=r'_{\text{ref}},\beta'=\frac{\pi}{2},\omega\right)$ can be decomposed as

$$\mathring{G}'_m\left(r'|_{r'=r'_{\text{ref}}},\beta'|_{\beta'=\frac{\pi}{2}},\omega\right) = = \sum_{n'=|m'|}^{\infty}\underbrace{\breve{G}'^{m'}_{n'}(\omega)\sqrt{\frac{2n'+1}{4\pi}\frac{(n'-m')!}{(n'+m')!}}\,P^{m'}_{n'}(0)}_{=\bar{G}'^{m'}_{n'}(\omega)}\, h^{(2)}_{n'}\left(\frac{\omega}{c}r'_{\text{ref}}\right) . \quad (23)$$

Again the factor $P^{m'}_{n'}(0)$ is apparent which is zero wherever $n'+m'$ is odd.
The right hand side of (23) constitutes an expansion of $\mathring{G}'_m\left(r'|_{r'=r'_{\text{ref}}},\beta'|_{\beta'=\frac{\pi}{2}},\omega\right)$ over spherical Hankel functions of fixed argument but different degrees n'. However, we are not aware that a method is readily available that allows to analytically extract the coefficients $\breve{G}'^{m'}_{n'}(\omega)$ via (23) from the circular microphone array measurements. Numerical methods such as [18] can be employed. However, their accuracy and limitations have not been investigated in detail.

It can therefore not be clarified in the scope of this paper whether a horizontal measurement is sufficient for the retrieval of the coefficients $\breve{G}'^{2k'-n'}_{n'}(\omega)$ or whether a three-dimensional measurement such as in [13] is necessary.

8. CONCLUSIONS

An approach for sound field reproduction employing circular arrangements of secondary sources was presented. It was focused on the general properties of the resulting secondary source driving function when non-omnidirectional secondary sources are used. In order that the presented approach is applicable the spatio-temporal characteristics of the employed secondary sources have to be invariant with respect to rotation around the center of the secondary source arrangement. In other words, all secondary sources have to exhibit equal radiation characteristics and have to be orientated towards the center of the secondary source arrangement.

Preliminary measurements of the ELAC 301 loudspeakers which are employed in the loudspeaker system installed at Deutsche Telekom Laboratories indicate that only very little variation in the spatio-temporal characteristics are apparent within different loudspeakers of the same model. This indicates that the presented approach is indeed applicable when all loudspeakers are of the same model. However, the investigation of resulting errors when such variation in the spatio-temporal characteristics of the secondary sources is apparent or when secondary sources are not properly positioned and orientated could not be included in the present paper.

It was shown that only a subset of the spherical harmonics coefficients of the loudspeaker directiviy have to be known. We assumed that these directivity coefficients are precisely known. This requires high resolution measurements of the coefficients in order to assure that no considerable spatial aliasing occurs. These measurements can be assumed to be less complex than in the conventional compensation approaches which require to measure the entire loudspeaker array (refer to Sec. 1).

It also advisable that the radius of the microphone array is not too different from the radius of the secondary source contour under consideration. The presented approach implicitly includes an extrapolation of the microphone array measurements to the radius of the secondary source contour. The restrictions of extrapolation of such spatially discrete data when spatial aliasing is apparent is not known.

Due to the fact that each spatial mode of the directivity filter can be pre-computed offline, it is likely that the precision requirements can be met. Future work includes an error analysis as described above.

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Appendix: Recursion formulae for singular-to-regular sectorial translation [14]

For $m \leq 0$:

$$b_{|m|+1}^{-|m|-1} \, (S|R)_{|m|+1,n'}^{-|m|-1,m'} \, (\Delta\mathbf{x},\omega) = \\ = b_{n'}^{m'} \, (S|R)_{|m|,n'-1}^{-|m|,m'+1} \, (\Delta\mathbf{x},\omega) - b_{n'+1}^{-m'-1} \, (S|R)_{|m|,n'+1}^{-|m|,m'+1} \, (\Delta\mathbf{x},\omega) \,, \tag{24}$$

for $m \geq 0$:

$$b_{m+1}^{-m-1} \, (S|R)_{m+1,n'}^{m+1,m'} \, (\Delta\mathbf{x},\omega) = \\ = b_{n'}^{-m'} \, (S|R)_{m,n'-1}^{m,m'-1} \, (\Delta\mathbf{x},\omega) - b_{n'+1}^{m'-1} \, (S|R)_{m,n'+1}^{m,m'-1} \, (\Delta\mathbf{x},\omega) \,, \tag{25}$$

with

$$b_n^m = \begin{cases} \sqrt{\frac{(n-m-1)(n-m)}{(2n-1)(2n+1)}} & \text{for } 0 \leq m \leq n \\ -\sqrt{\frac{(n-m-1)(n-m)}{(2n-1)(2n+1)}} & \text{for } -n \leq m < 0 \\ 0 & \text{for } |m| > n \,. \end{cases} \tag{26}$$